Differentiation in Banach spaces

link: http://www.cs.utk.edu/~mclennan/Classes/594-MNN/

Fréchet derivative

Definition 2.4.1 (Fréchet differentiation) *Suppose X and Y are two Banach spaces and U is an open subset of X. Then $T : U \to Y$ is* Fréchet differentiable *at ϕ if there is a bounded linear operator $D : X \to Y$ such that the following holds. For all $\alpha \in X$ such that $\phi + \alpha \in U$, there is an $E : X \to Y$ such that*

$$T(\phi + \alpha) = T(\phi) + D(\alpha) + E(\alpha)$$

and

$$\lim_{\|\alpha\| \to 0} \frac{\|E(\alpha)\|}{\|\alpha\|} = 0.$$

Under these circumstances, D is called the Fréchet derivative of T at ϕ; it is denoted by $T'(\phi)$. The Fréchet derivative is a locally linear approximation to T; $T'(\phi)(\alpha) = D(\alpha)$ is called the Fréchet differential *of T.*

Remark 2.4.1 *Since a linear operator is continuous if and only if it is bounded, Fréchet derivatives are (by definition) continuous.*

Proposition 2.4.1 *The derivative of a linear operator is that operator: $L'(\phi) = L$.*

Gâteaux derivative

Remark 2.4.2 *Note that $T' : X \to \mathcal{L}(X, Y)$, where $\mathcal{L}(X, Y)$ is the space of all continuous (bounded) linear operators from X to Y.*

Remark 2.4.3 *Higher order derivatives are defined in the obvious way. Suppose $T : X \to Y$. Since $T' : X \to \mathcal{L}(X, Y)$, it is easy to see that the higher derivatives have the types:*

$$T'' : X \to \mathcal{L}(X, \mathcal{L}(X, Y)),$$
$$T^{(3)} : X \to \mathcal{L}(X, \mathcal{L}(X, \mathcal{L}(X, Y))),$$

and so forth. Note that each successive derivative is of "higher type" than its predecessor.

Definition 2.4.2 (Gâteaux differentiation) *Suppose X and Y are Banach spaces, $U \subseteq X$ is open, and $T : U \to Y$. Then T has a Gâteaux derivative at $\phi \in U$ if, for all $\alpha \in U$ the following limit exists:*

$$\mathrm{d}T(\phi, \alpha) = \lim_{t \to 0} \frac{T(\phi + t\alpha) - T(\phi)}{t} = \frac{\mathrm{d}}{\mathrm{d}t} T(\phi + t\alpha)\big|_{t=0}.$$

We write $\mathrm{d}T(\phi, \alpha)$ for the Gâteaux derivative of T at ϕ in the "direction" α.

Proposition 2.4.2 *The Gâteaux derivative, if it exists, is unique.*