if $|\kappa| \leq b$ and if $\psi, H\psi, \ldots, H^m\psi \in D(\exp(ar))$ for suitable m. Since $\hat{\psi}(\cdot + i\kappa)$ is the Fourier transform of $e^{\kappa \cdot \zeta}\psi(\zeta)$, (70) implies that

$$|e^{\kappa \cdot \zeta}\psi(\zeta)| \leq D_b(2\pi)^{-(3N-3)/2}$$

if $|\kappa| \leq b$. Taking a sup over such κ we see that

$$|e^{br}\psi(\zeta)| \leq D_b(2\pi)^{-(3N-3)/2} \quad \blacksquare$$

Corollary Let H be an atomic Hamiltonian. Then any eigenfunction ψ corresponding to a point in the discrete spectrum obeys an estimate of the form

$$|\psi(\zeta)| \leq C_a e^{-ar}$$

for some positive a.

XIII.12 Nondegeneracy of the ground state

Our main result in this section will concern a self-adjoint operator H that is bounded below and has an eigenvalue as the lowest point in its spectrum. Under certain conditions, we shall show that the eigenspace corresponding to that eigenvalue is one dimensional and that the corresponding eigenvector is a strictly positive function in a particular realization of the underlying Hilbert space as an L^2 space. The eigenvalue is called the **ground state energy** and the eigenvector the **ground state**. As is the case with so much spectral analysis, the method of verifying the conditions in applications is a perturbative technique. At first sight this is surprising since we shall be interested in Schrödinger operators and $H_0 = -\Delta$ does not possess any eigenfunctions! It will thus be important that all our results will be in terms of conclusions of the form: "Either H has no eigenvalue below the essential spectrum or it has a nondegenerate ground state energy with associated eigenvector strictly positive." Finally, we note that, as usual, it is easier to study the bounded operators e^{-tH} and $(H+1)^{-1}$ rather than H. If E_0 is the ground state energy for H, then e^{-tE_0} is the *largest* eigenvalue of e^{-tH}. We begin by studying a bounded operator A and its largest eigenvalue.

Definition Let $\langle M, \mu\rangle$ be a σ-finite measure space. $f \in L^2(M, d\mu)$ is called **positive** if f is nonnegative a.e. and is not the zero function. f is called **strictly positive** if $f(x) > 0$ a.e.. A bounded operator A on L^2 is called **positivity preserving** if Af is positive whenever f is positive. A is called **positivity**

improving if Af is strictly positive whenever f is positive. Finally, A is called **ergodic** if and only if it is positivity preserving and for any $u, v \in L^2$ that are both positive there is some $n > 0$ for which $(u, A^n v) \neq 0$.

We remark that by definition the zero function is not positive; so if A is positivity preserving, then Af is not the zero function for any positive function f. Moreover, by the following reformulation, every positivity improving map is ergodic: A function $g \in L^2(M, d\mu)$ is strictly positive if and only if $(f, g) > 0$ for all positive functions f. Thus, a bounded operator A on $L^2(M, d\mu)$ is positivity improving if and only if $(u, Av) > 0$ for all positive functions $u, v \in L^2(M, d\mu)$.

Example 1 Let $A = (-\Delta + 1)^{-1}$ on $L^2(\mathbb{R}^3)$. Equation (IX.30) states that

$$(f, Ag) = \int \overline{f(x)} g(y) \frac{e^{-|x-y|}}{4\pi|x-y|} d^3x\, d^3y$$

Since $(4\pi|x-y|)^{-1} e^{-|x-y|}$ is strictly positive, A is positivity improving. Similarly, the explicit formula for $e^{t\Delta}$ on $L^2(\mathbb{R}^n)$ shows that it is also positivity improving.

Example 2 Let $\langle M, \mu \rangle$ be a probability measure space, i.e., $\mu(M) = 1$ and let $T: M \to M$ be a measure preserving map. Let $(Af)(x) = f(Tx)$. A is a unitary operator that is clearly positive preserving. It is never positivity improving. A is ergodic if and only if T is ergodic in the sense defined in Section II.5 (Problem 90).

Example 3 (second quantized operators on a Fock space) We saw in the proof of Theorem X.61 (Section X.9) that $\Gamma(e^{-tA})$ is positivity preserving as an operator on Q space whenever A is a positive self-adjoint operator on the one-particle space that commutes with the distinguished complex conjugation. We shall examine later the question of when it is ergodic or positivity improving.

The main abstract theorem of this section is the following:

Theorem XIII.43 Let A be a bounded, positive operator on $L^2(M, d\mu)$. Suppose that A is positivity preserving and that $\|A\|$ is an eigenvalue. Then the following are equivalent:

(a) $\|A\|$ is a simple eigenvalue and the associated eigenvector is strictly positive.

(b) A is ergodic.
(c) $L^\infty(M) \cup \{A\}$ acts irreducibly, i.e., no nontrivial closed subspace is left invariant by both A and every bounded multiplication operator.

Proof We prove (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (a).

(a) $\Rightarrow$ (b) Let $B = A/\|A\|$ and let $\{P_\Omega\}$ be the spectral projections for B. Since $x^n \to 0$ if $0 \leq x < 1$ and $x^n \to 1$ if $x = 1$, the functional calculus implies that

$$\operatorname*{s-lim}_{n\to\infty} B^n = P_{\{1\}}$$

By hypothesis, $P_{\{1\}} = (\psi, \cdot)\psi$ where ψ is strictly positive. Thus for any positive u, v,

$$\lim_{n\to\infty} (u, B^n v) = (u, \psi)(\psi, v) > 0$$

As a result, for some n, $(u, A^n v) = \|A\|^n (u, B^n v) > 0$.

(b) $\Rightarrow$ (c) Suppose (c) does not hold. Let S be a nontrivial subspace left invariant by $L^\infty(M) \cup \{A\}$. Let $f \in S$ and let $h = \bar{f}/|f| \in L^\infty(M)$. Then $|f| = hf \in S$. Similarly, if $g \in S^\perp$, then $|g| \in S^\perp$. Choose $f \in S$, $g \in S^\perp$, $f \neq 0 \neq g$. Since A leaves S invariant, $A^n|f| \in S$, so

$$(|g|, A^n|f|) = 0$$

for all n. Thus A is not ergodic.

(c) $\Rightarrow$ (a) By hypothesis, $\|A\|$ is an eigenvalue. Let ψ be an eigenvector of A with eigenvalue $\|A\|$ and suppose that ψ is real valued. Since $|\psi| \pm \psi \geq 0$, we know that $A(|\psi| \pm \psi) \geq 0$ so

$$|A\psi| \leq A|\psi|$$

Thus

$$(|\psi|, A|\psi|) \geq (|\psi|, |A\psi|) \geq (\psi, A\psi) = \|A\| \, \|\psi\|^2$$

As a result, $A|\psi| = \|A\| \, |\psi|$ so $|\psi|$ is also an eigenvector. We shall show that $|\psi|$ is strictly positive. Let $S = \{f \in L^2(M, d\mu) \mid f\psi = 0 \text{ a.e.}\}$. S is clearly a subspace and moreover it is left invariant by $L^\infty(M)$. Let $S_+ = \{g \in S \mid g \geq 0\}$. Then for $f \in S_+$, $(Af, |\psi|) = (f, A|\psi|) = \|A\|(f, |\psi|) = 0$. Since Af is positive, $(Af)\psi = 0$ a.e., that is, $Af \in S_+$. Thus, A leaves S_+ invariant. Since $S = S_+ - S_+ + i(S_+ - S_+)$, A leaves S invariant. Thus by hypothesis (c), $S = \{0\}$ or $S = \mathscr{H}$. Since $\psi \notin S$ and $\psi \neq 0$, $S = \{0\}$, from which it follows that $|\psi| > 0$ a.e.

Thus we know that any real eigenvector with eigenvalue $\|A\|$ is nonzero a.e. and that also $A|\psi| = \|A\| \, |\psi|$. Thus $|\psi| - \psi$ is either an eigenvector with eigenvalue $\|A\|$ or it is identically 0 so that $|\psi| - \psi$ is either almost

everywhere vanishing or almost everywhere nonvanishing. This means that every real eigenvector with eigenvalue $\|A\|$ is a.e. strictly positive or a.e. strictly negative. If A had two real eigenvectors with eigenvalue $\|A\|$, it would have two orthogonal eigenvectors which were both a.e. positive. Since this is impossible, we conclude that A has only one real eigenvector of eigenvalue $\|A\|$ and that this eigenvector is strictly positive.

Finally, let ψ be an arbitrary eigenvector with eigenvalue $\|A\|$. Since A takes positive functions into positive functions, it takes real functions into real functions, so that $\mathrm{Re}(A\psi) = A(\mathrm{Re}\,\psi)$. Thus $\mathrm{Re}\,\psi$ and $\mathrm{Im}\,\psi$ are eigenvectors with eigenvalue $\|A\|$. It follows that ψ is a complex multiple of the unique real eigenvector. ∎

To apply this to the lowest eigenvalue of a self-adjoint operator, we can use either the resolvent or the semigroup generated by the operator. We first note:

Proposition Let H be a self-adjoint operator that is bounded from below. Let $E = \inf \sigma(H)$. Then e^{-tH} is positivity preserving for *all* $t > 0$ if and only if $(H - \lambda)^{-1}$ is positivity preserving for *all* $\lambda < E$.

Proof The proposition follows from the formulas

$$(H - \lambda)^{-1}\varphi = \int_0^\infty e^{\lambda t} e^{-Ht}\varphi \, dt \qquad (\lambda < E)$$

$$e^{-tH}\varphi = \lim_{n\to\infty} \left(1 + \frac{tH}{n}\right)^{-n} \varphi \qquad (t > 0) \tag{71}$$

These formulas, which are familiar from the theory of semigroups, (Section X.8) can be proven easily in the self-adjoint case by the functional calculus.

Theorem XIII.44 Let H be a self-adjoint operator that is bounded from below on $L^2(M, d\mu)$. Suppose that e^{-tH} is positivity preserving for all $t > 0$ and that $E = \inf \sigma(H)$ is an eigenvalue. Then the following are equivalent:

(a) E is a simple eigenvalue with a strictly positive eigenvector.
(b) $(H - \lambda)^{-1}$ is ergodic for some $\lambda < E$.
(c) e^{-tH} is ergodic for some $t > 0$.
(d) $(H - \lambda)^{-1}$ is positivity improving for all $\lambda < E$.
(e) e^{-tH} is positivity improving for all $t > 0$.

Proof By Theorem XIII.43, (a), (b), (c) are equivalent and clearly (d) ⇒ (b) and (e) ⇒ (c). We complete the proof by showing that (c) ⇒ (d) and (e).

$(c) \Rightarrow (d)$ Let u and v be positive. Since some e^{-tH} is ergodic, $(u, e^{-sH}v) > 0$ for some $s > 0$. But $(u, e^{-sH}v)$ is continuous so $(u, e^{-sH}v) > 0$ for s in some interval. Thus

$$(u, (H - \lambda)^{-1}v) = \int_0^\infty e^{\lambda s}(u, e^{-sH}v)\, ds > 0$$

$(c) \Rightarrow (e)$ Let u, v be positive and let $B = \{t > 0 \mid (u, e^{-tH}v) > 0\}$. Since B is nonempty and $(u, e^{-tH}v)$ is analytic in a neighborhood of the positive real axis, $(0, \infty)\backslash B$ can have only 0 as a limit point. In particular B contains arbitrarily small numbers. Thus, if we can show that $t > s$ and $s \in B$ implies $t \in B$, we can conclude that $B = (0, \infty)$. Fix $s \in B$. Then $(u, e^{-sH}v) > 0$ so that $u(\cdot)(e^{-sH}v)(\cdot)$ is not identically zero. Let $w = \min\{u, e^{-sH}v\}$. Then $w(m)$ is not identically zero. Since $e^{-\tau H}$ is positivity preserving,

$$\begin{aligned}(u, e^{-\tau H}(e^{-sH}v)) &\geq (u, e^{-\tau H}w) = (e^{-\tau H}u, w) \\ &\geq (e^{-\tau H}w, w) = \|e^{-\tau H/2}w\|^2 > 0\end{aligned}$$

We have used the fact that w is positive and that $e^{-\tau H/2}$ is positivity preserving to conclude that $e^{-\tau H/2}w \neq 0$. Thus $s \in B$ and $\tau > 0$ imply $s + \tau \in B$. ∎

Example 3, revisited Let A be an operator on a complex Hilbert space $\mathscr{H}$ that commutes with a distinguished complex conjugation and which obeys $A \geq cI$ for some $c > 0$. Let $H = d\Gamma(A)$ be the second quantization of A on $\mathscr{F}_s(\mathscr{H})$ thought of as an operator on $L^2(Q, d\mu)$ (see Section X.7). Then $H\Omega_0 = 0$ while $A \restriction \{\Omega_0\}^\perp \geq cI$. Thus H has a nondegenerate strictly positive ground state. We conclude that $\Gamma(e^{-tA}) = e^{-tH}$ is positivity improving for all $t > 0$.

In applications of Theorem XIII.44, the following perturbation theorem is very useful. There is another perturbation result based on the resolvent rather than the semigroup which is also useful (see Problems 91, 92).

Theorem XIII.45 Let H and H_0 be semibounded, self-adjoint operators on $L^2(M, d\mu)$ where $\langle M, \mu\rangle$ is a σ-finite measure space. Suppose that there exists a sequence of bounded multiplication operators V_n so that $H_0 + V_n$ converges to H in strong resolvent sense and so that $H - V_n$ converges to H_0 in strong resolvent sense. Suppose, moreover, that $H - V_n$ and $H_0 + V_n$ are uniformly bounded from below. Then:

(a) e^{-tH} is positivity preserving if and only if e^{-tH_0} is positivity preserving.
(b) $\{e^{-tH}\} \cup L^\infty(M, d\mu)$ acts irreducibly on $L^2(M, d\mu)$ if and only if $\{e^{-tH_0}\} \cup L^\infty(M, d\mu)$ acts irreducibly on $L^2(M, d\mu)$.

Proof By the Trotter product formula (Theorem VIII.31) and the continuity of the functional calculus (Theorem VIII.20),

$$e^{-tH} = \text{s-lim}_{n\to\infty}\left(\text{s-lim}_{m\to\infty}\left[e^{-tH_0/m}e^{-tV_n/m}\right]^m\right) \tag{72a}$$

$$e^{-tH_0} = \text{s-lim}_{n\to\infty}\left(\text{s-lim}_{m\to\infty}\left[e^{-tH/m}e^{+tV_n/m}\right]^m\right) \tag{72b}$$

Since $e^{\pm tV_n/m}$ is positivity preserving, we see that (a) holds. Moreover, (72) and the fact that $e^{\pm tV_n/m} \in L^\infty(M)$ imply that any subspace left invariant by e^{-tH_0} and $L^\infty(M)$ is left invariant by e^{-tH}. ∎

We can now apply Theorem XIII.44 to Schrödinger operators and to spatially cut-off $P(\varphi)_2$ Hamiltonians.

Theorem XIII.46 Let H be the Hamiltonian of an N-body Schrödinger system with center of mass motion removed. Suppose that the potentials V_{ij} are in $R + (L^\infty)_\varepsilon$. Then, if H has an eigenvalue at the bottom of its spectrum, the eigenvalue is nondegenerate and the corresponding eigenfunction is strictly positive.

Proof By Theorems XIII.43 and XIII.44, we need only prove that e^{-tH} is positivity preserving and that $\{e^{-tH}\} \cup L^\infty(\mathbb{R}^{3N-3})$ acts irreducibly. By Example 1 and the proof (b) ⇒ (c) in Theorem XIII.43, we know these facts for e^{-tH_0}. Let $V_{ij}^{(n)}$ be defined by

$$V_{ij}^{(n)}(x) = \begin{cases} V_{ij}(x) & \text{if } |V_{ij}(x)| \le n \\ n & \text{if } V_{ij}(x) > n \\ -n & \text{if } V_{ij}(x) < -n \end{cases}$$

It is easy to prove that $V_{ij} - V_{ij}^{(n)} \to 0$ in the Rollnik norm. By Theorem VIII.25c, $H_0 + \sum V_{ij}^{(n)} \to H$ and $H - \sum V_{ij}^{(n)} \to H_0$ in norm resolvent sense. Thus, Theorem XIII.45 is applicable and the proof is completed. ∎

Corollary Let H satisfy the hypothesis of Theorem XIII.46 and suppose that $H\psi = E\psi$ where $E = \inf \sigma(H)$. Let U be any positivity preserving unitary operator commuting with H. Then $U\psi = \psi$.

Proof Since U commutes with H, $H(U\psi) = E(U\psi)$. But E is nondegenerate so $U\psi = \alpha\psi$. Since U is unitary, $|\alpha| = 1$ and since U is positivity preserving and ψ is positive, $\alpha = 1$. ∎

For example, if the V_{ij} are all spherically symmetric and H has an eigenvalue at the bottom of its spectrum, that eigenvector is left invariant by rotations and reflections.

We warn the reader that Theorem XIII.46 deals with the lowest eigenvalue of H as an operator on all of $L^2(\mathbb{R}^{3N-3})$. If H commutes with a family $\mathscr{P}$ of permutations of the coordinates and we restrict to a subspace of $L^2(\mathbb{R}^{3N-3})$ with a certain permutation symmetry, then the lowest eigenvalue of the restricted operator need not be nondegenerate. In fact, by the corollary, unless we look at the subspace left *pointwise* invariant by $\mathscr{P}$, the ground state of the restricted operator is an excited state of the unrestricted operator. This implies the statement that *ground states of Boson systems are nondegenerate while ground states of Fermi systems may be degenerate.*

Theorem XIII.47 Let $V \in L^2_{\text{loc}}(\mathbb{R}^m)$ be positive and suppose $\lim_{|x|\to\infty} V(x) = \infty$. Then $-\Delta + V$ has a nondegenerate strictly positive ground state.

Proof We shall see in Section 14 that $-\Delta + V$ has purely discrete spectrum so the bottom of its spectrum is certainly an eigenvalue. The proof of the remaining conclusions follows closely that of Theorem XIII.46 with one detail different. Let $V_n = \min\{V, n\}$. Then $-\Delta + V_n$, $-\Delta + V$, $-\Delta$, and $-\Delta + (V - V_n)$ are all essentially self-adjoint on $C_0^\infty(\mathbb{R}^m)$ by Theorem X.28. Moreover, for any $\psi \in C_0^\infty(\mathbb{R}^m)$, $V_n\psi \to V\psi$ in L^2. Thus by Theorem VIII.25a we have the necessary strong resolvent convergence. ∎

If V is allowed to be very singular, then $-\Delta + V$, defined as a sum of quadratic forms, can have a degenerate ground state (see Example 2 in Section 13 and Problem 95). A reference for the following result is given in the Notes.

Theorem XIII.48 Let G be a closed subset of $\mathbb{R}^n$ of measure zero and suppose that $V \in L^1_{\text{loc}}(\mathbb{R}^n \backslash G)$ with $V \geq 0$. Let $H = -\Delta + V$ be defined as the sum of quadratic forms. Then:

(a) If $\mathbb{R}^n \backslash G$ is connected and $\inf \sigma(H)$ is an eigenvalue, then it is a simple eigenvalue and the corresponding eigenfunction is strictly positive.
(b) If $\mathbb{R}^n \backslash G$ is disconnected, V can be chosen so that H has a degenerate ground state.

Finally, we turn to existence and uniqueness of ground states in the framework of hypercontractive semigroups (see Section X.9):

Theorem XIII.49 Let H_0 generate a positivity improving hypercontractive semigroup on a probability measure space $\langle M, \mu\rangle$. Suppose that V is a (not necessarily bounded) multiplication operator with V and e^{-V} in $\bigcap_{p<\infty} L^p(M, d\mu)$. Let $H = H_0 + V$. Then:

(a) $E = \inf \sigma(H)$ is an eigenvalue.
(b) E is nondegenerate.
(c) The ground state eigenfunction is strictly positive.

In particular, (a)–(c) hold for the spatially cutoff $P(\varphi)_2$ Hamiltonians.

Proof We shall first prove (b) and (c) assuming (a) and then prove (a). Let V_n be defined by

$$V_n(q) = \begin{cases} V(q) & \text{if } |V(q)| \leq n \\ n & \text{if } V(q) > n \\ -n & \text{if } V(q) < -n \end{cases}$$

Since $V_n \to V$ in each L^p $(p < \infty)$ and $\sup_n \{\|e^{-V_n}\|_p, \|e^{-(V-V_n)}\|_p\} < \infty$, Theorem X.60 says that $H_0 + V_n \to H_0 + V$ and $H_0 + V - V_n \to H_0$ in norm resolvent sense. Since e^{-tH_0} is a positivity improving semigroup, Theorem XIII.45 is applicable and thus (b) and (c) hold if (a) is true.

To prove (a), we proceed as follows. Let t be chosen so that $A = \exp(-tH)$ is bounded from L^2 to L^4. We need only prove that $\|A\|$ is an eigenvalue of A. A **finite partition** of M is a family $\{S_1, \ldots, S_n\} = \alpha$ of measurable subsets of M that are mutually disjoint, and whose union is M. If α and β are partitions, we write $\beta > \alpha$ if each set in β is a subset of a set in α. For each partition α, define the projection

$$P_\alpha = \sum_{S \in \alpha} \mu(S)^{-1}(\chi_S, \cdot)\chi_S$$

and let $A_\alpha = P_\alpha A P_\alpha$. We first prove the following properties of $\{P_\alpha\}$:

(i) s-$\lim_\alpha P_\alpha = 1$.
(ii) P_α is positivity preserving.
(iii) P_α is a contraction on each $L^p(M, d\mu)$.

To prove (i), we note that for any simple function ψ, $P_\alpha \psi = \psi$ for α sufficiently large. Since the simple functions are dense, (i) follows. (ii) is obvious, and (iii) follows from (ii), $P_\alpha 1 = 1$, and Theorem X.55a. Thus $\{A_\alpha\}$ obeys:

(iv) s-$\lim_\alpha A_\alpha = A$.
(v) $\|A\| = \lim_\alpha \|A_\alpha\|$.
(vi) There is a K with $\|A_\alpha \psi\|_4 \leq K\|A_\alpha\| \, \|\psi\|_2$ for all $\psi \in L^2$ and all α.

(iv) follows from (i). (vi) follows from (iii) and the fact that $\|A\psi\|_4 \leq \tilde{K}\|\psi\|_2$ for some $\tilde{K}$ and the bound $\|A_\alpha\| \geq$ (A1, 1). To prove (v), notice first that $\|A_\alpha\| \leq \|A\|$ since $\|P_\alpha\| = 1$. On the other hand, by (iv), $\|A\| \leq \underline{\lim}\|A_\alpha\|$. Thus $\lim\|A_\alpha\|$ exists and equals $\|A\|$.

Fix a partition $\alpha = \{S_1, \ldots, S_n\}$. Then Ran P_α is naturally isomorphic to $\mathbb{C}^n$ via the map $\sum c_i \chi_{S_i} \mapsto (c_1, \ldots, c_n)$. View $\mathbb{C}^n$ as $L^2(\{1, \ldots, n\}, d\nu)$ with $\nu(i) = \mu(S_i)$. Then A_α is positivity preserving on $\mathbb{C}^n$. Since A_α is a positive definite finite-dimensional matrix, $\|A_\alpha\|$ is an eigenvalue, so we can find $\varphi_\alpha \in$ Ran P_α with $A_\alpha\varphi_\alpha = \|A_\alpha\|\varphi_\alpha$. By the argument in the proof of Theorem XIII.43, $\psi_\alpha = |\varphi_\alpha|$ is an eigenvector. Normalize ψ_α so that $\|\psi_\alpha\|_2 = 1$. Then, on the one hand, $\|\psi_\alpha\|_4 \leq K$, and on the other $1 = \|\psi_\alpha\|_2 \leq \|\psi_\alpha\|_1^{1/3}\|\psi_\alpha\|_4^{2/3}$ by Hölder's inequality. Thus $\|\psi_\alpha\|_1 \geq K^{-2}$. Let ψ be a weak limit point of the net $\{\psi_\alpha\}$, and let $\{\psi_\beta\}$ be a subnet converging to ψ. Then for any $\varphi \in L^2$,

$$(\varphi, A\psi) = (A\varphi, \psi) = \lim_\beta (A\varphi, \psi_\beta) = \lim_\beta \|A_\beta\|(\varphi, \psi_\beta) = \|A\|(\varphi, \psi)$$

so $A\psi = \|A\|\psi$. Moreover, since $\psi_\beta \geq 0$, $\|\psi_\alpha\|_1 = (1, \psi_\alpha) \geq K^{-2}$. Thus $(1, \psi) \geq K^{-2}$. In particular, $\psi \neq 0$. Thus $\|A\|$ is an eigenvalue. ∎

Appendix 1 to Section XIII.12 The Beurling-Deny criteria

There is an especially simple criterion for a positive self-adjoint operator H on $L^2(M, d\mu)$ to generate a positivity preserving semigroup.

Theorem XIII.50 (first Beurling–Deny criterion) Let $H \geq 0$ be a self-adjoint operator on $L^2(M, d\mu)$. Extend $(\psi, H\psi)$ to all of L^2 by setting it equal to infinity when $\psi \notin Q(H)$. Then the following are equivalent:

(a) e^{-tH} is positivity preserving for all $t > 0$.
(b) $(|u|, H|u|) \leq (u, Hu)$ for all $u \in L^2$.
(c) e^{-tH} is reality preserving and

$$(u_+, Hu_+) \leq (u, Hu)$$

for all real-valued $u \in L^2$. Here $u_+(x) \equiv \max\{u(x), 0\}$.
(d) e^{-tH} is reality preserving and

$$(u_+, Hu_+) + (u_-, Hu_-) \leq (u, Hu)$$

for all real-valued $u \in L^2$. Here $u_- = u_+ - u$.