4 Angular momentum – useful formulae and results

In this appendix, we collect, for the most part without proof, useful relations and results concerned with angular momentum. A good elementary treatment of angular momentum can be found in the texts by Powell and Crasemann (1962) and Merzbacher (1970) while a more advanced and complete treatment can be found in the monographs of Edmonds (1957), and Rose (1957).

Angular momentum operators

In Chapter 2, we discussed the representation of the components of the orbital angular momentum **L** by differential operators, starting from the expression $\mathbf{L} = \mathbf{r} \times \mathbf{p}$. We also outlined the theory of particles of spin one-half, for which the components of the spin angular momentum **S** were represented by 2×2 matrices. We shall obtain a general matrix representation for the Cartesian components J_x, J_y and J_z of the angular momentum operator **J**, which will include the particular cases such that the angular momentum is purely of orbital or of spin type.

The operators J_x, J_y and J_z are *defined* as linear self-adjoint operators satisfying the commutation relations

$$[J_x, J_y] = i\hbar J_z \text{ (and cyclicly)} \qquad [A4.1]$$

Since $\mathbf{J}^2 = J_x^2 + J_y^2 + J_z^2$ commutes with each component, simultaneous eigenfunctions ψ_{jm} of $\mathbf{J}^2$ and one component, say J_z, can be found, with

$$\begin{aligned} \mathbf{J}^2\psi_{jm} &= j(j+1)\hbar^2\psi_{jm} \\ J_z\psi_{jm} &= m\hbar\psi_{jm} \end{aligned} \qquad [A4.2]$$

We may normalise the ψ_{jm} to unity, in which case the orthonormality relations

$$\langle\psi_{j'm'}|\psi_{jm}\rangle = \delta_{jj'}\,\delta_{mm'} \qquad [A4.3]$$

are satisfied, and J_z is represented by a diagonal matrix with elements

$$\langle\psi_{j'm'}|J_z|\psi_{jm}\rangle = \delta_{jj'}\,\delta_{mm'}m\hbar \qquad [A4.4]$$

In this matrix representation, the eigenfunctions ψ_{jm} are, in fact, column vectors.

The eigenvalues, which we have written for later convenience in the form $j(j+1)\hbar^2$ and $m\hbar$, are real as $\mathbf{J}^2$ and $\mathcal{J}_z$ are self-adjoint. They can be determined by the following argument, which we give in outline only.

Let us define the raising and lowering operators $\mathcal{J}_\pm$ as

$$\mathcal{J}_\pm = \mathcal{J}_x \pm i\mathcal{J}_y \qquad \text{[A4.5]}$$

where $\mathcal{J}_+ = \mathcal{J}_-^\dagger$; $\mathcal{J}_- = \mathcal{J}_+^\dagger$. We note the relations

$$[\mathcal{J}_z, \mathcal{J}_\pm] = \pm\hbar\mathcal{J}_\pm \qquad \text{[A4.6]}$$

$$[\mathbf{J}^2, \mathcal{J}_\pm] = 0 \qquad \text{[A4.7]}$$

$$\mathcal{J}_+\mathcal{J}_- = \mathbf{J}^2 - \mathcal{J}_z^2 + \hbar\mathcal{J}_z; \qquad \mathcal{J}_-\mathcal{J}_+ = \mathbf{J}^2 - \mathcal{J}_z^2 - \hbar\mathcal{J}_z \qquad \text{[A4.8]}$$

From the commutation relation [A4.6], we have

$$\begin{aligned}\mathcal{J}_z\{\mathcal{J}_\pm\psi_{jm}\} &= \mathcal{J}_\pm\mathcal{J}_z\psi_{jm} \pm \hbar\mathcal{J}_\pm\psi_{jm} \\ &= (m \pm 1)\hbar\{\mathcal{J}_\pm\psi_{jm}\}\end{aligned} \qquad \text{[A4.9]}$$

so that $(\mathcal{J}_\pm\psi_{jm})$ are eigenfunctions of $\mathcal{J}_z$ belonging to the eigenvalues $(m \pm 1)$. Because of [A4.7], these functions are simultaneously eigenfunctions of $\mathbf{J}^2$, belonging to the eigenvalue $j(j+1)\hbar^2$.

For any wave function ϕ, $\langle\phi|\mathbf{J}^2|\phi\rangle \geqslant \langle\phi|\mathcal{J}_z^2|\phi\rangle$, and setting $\phi = \psi_{jm}$, we find

$$j(j+1) \geqslant m_j^2 \qquad \text{[A4.10]}$$

By operating with $\mathcal{J}_+$ or $\mathcal{J}_-$ repeatedly, sequences of eigenfunctions of $\mathcal{J}_z$ can be constructed, namely $(\mathcal{J}_+)^n\psi_{jm}$, $(\mathcal{J}_-)^{n'}\psi_{jm}$, with eigenvalues $(m+n)\hbar$ and $(m-n')\hbar$ respectively. In view of [A4.10], for each j there must be a maximum eigenvalue of $\mathcal{J}_z$, say $\lambda\hbar$, and also a minimum eigenvalue, say $\lambda'\hbar$, such that $\lambda - \lambda'$ = an integer (or zero). If $\mathcal{J}_+$ is applied to $\psi_{j\lambda}$, we must have $\mathcal{J}_+\psi_{j\lambda} = 0$, for otherwise the sequence would not terminate and $\lambda\hbar$ would not be the maximum eigenvalue. Using [A4.8], we have

$$\mathcal{J}_-\mathcal{J}_+\psi_{j\lambda} = \{j(j+1) - \lambda^2 - \lambda\}\hbar^2\psi_{j\lambda} = 0 \qquad \text{[A4.11]}$$

with the solution $\lambda = j$. In the same way $\mathcal{J}_-\psi_{j\lambda'} = 0$, from which we find $\lambda' = -j$. Since $(\lambda - \lambda')$ is an integer (or zero), $(2j)$ is an integer (or zero) and j must be one of the integers or half-integers, $j = 0, \frac{1}{2}, 1, 3/2, 2, \ldots$. For a given value of j, m can take the $(2j+1)$ values $-j, -j+1, \ldots j-1, j$.

To find the matrix elements of $\mathcal{J}_x$ and $\mathcal{J}_y$, or equivalently of $\mathcal{J}_+$ and $\mathcal{J}_-$, we note that

$$\mathcal{J}_+\psi_{jm} = N\psi_{jm+1} \qquad \text{[A4.12]}$$

where N is a constant. Since both ψ_{jm} and ψ_{jm+1} are normalised to unity, we have from [A4.3]

$$\begin{aligned}N^2 &= \langle\mathcal{J}_+\psi_{jm}|\mathcal{J}_+\psi_{jm}\rangle = \langle\psi_{jm}|\mathcal{J}_-\mathcal{J}_+|\psi_{jm}\rangle \\ &= \hbar^2(j(j+1) - m(m+1))\end{aligned} \qquad \text{[A4.13]}$$

where we have used [A4.8]. Adopting the convention that N is real and positive, we then obtain

$$N = \hbar \sqrt{j(j+1) - m(m+1)} \quad \text{[A4.14]}$$

From [A4.12], the matrix representing $\mathcal{J}_+$ in the basis of eigenfunctions ψ_{jm} is

$$\langle \psi_{j'm'} | \mathcal{J}_+ | \psi_{jm} \rangle = \sqrt{j(j+1) - m(m+1)}\hbar \, \delta_{jj'} \, \delta_{m'm+1} \quad \text{[A4.15]}$$

In a similar way, we find

$$\langle \psi_{j'm'} | \mathcal{J}_- | \psi_{jm} \rangle = \sqrt{j(j+1) - m(m-1)}\hbar \, \delta_{jj'} \, \delta_{m'm-1} \quad \text{[A4.16]}$$

As we saw in Chapter 2, if $\mathbf{J}$ is a pure orbital angular momentum ($\mathbf{L}$), the wave function must be single valued as a function of position, and this excludes the half-integral values of j. For a spin angular momentum or when $\mathbf{J}$ is the sum of an orbital and a spin angular momentum, both the integral and half-integral values are allowed.

Spherical harmonics and Legendre polynomials

In Chapter 2 we introduced the spherical harmonics $Y_{lm}(\theta, \phi)$, which are the simultaneous eigenfunctions of the orbital angular momentum operators $\mathbf{L}^2$ and L_z,

$$\mathbf{L}^2 Y_{lm} = l(l+1)\hbar^2 Y_{lm}; \qquad L_z Y_{lm} = m\hbar Y_{lm} \quad \text{[A4.17]}$$

where $l = 0, 1, 2, \ldots$ and $m = -l, -l+1, \ldots l-1, l$. They satisfy the orthonormality relation

$$\int Y^\star_{l'm'}(\theta, \phi) Y_{lm}(\theta, \phi) \, \mathrm{d}\Omega = \delta_{ll'} \, \delta_{mm'}; \qquad (\mathrm{d}\Omega = \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi) \quad \text{[A4.18]}$$

and the closure relation

$$\sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y^\star_{lm}(\theta, \phi) Y_{lm}(\theta', \phi') = \delta(\Omega - \Omega') \quad \text{[A4.19]}$$

where

$$\delta(\Omega - \Omega') = \frac{1}{\sin\theta} \delta(\theta - \theta') \, \delta(\phi - \phi')$$

If the operators $L_\pm$ are defined as $L_\pm = L_x \pm iL_y$ (see [A4.5]) we have

$$L_\pm = \hbar e^{\pm i\phi} \left[\pm \frac{\partial}{\partial\theta} + i \cot\theta \frac{\partial}{\partial\phi} \right] \quad \text{[A4.20]}$$

and

$$L_\pm Y_{lm} = \hbar \sqrt{l(l+1) - m(m \pm 1)} \, Y_{lm\pm 1} \quad \text{[A4.21]}$$

in agreement with the general results of [A4.15] and [A4.16].

In the special case $m = 0$, the spherical harmonics are given by

$$Y_{l0} = \sqrt{\frac{2l + 1}{4\pi}}\, P_l(\cos\theta) \qquad [A4.22]$$

where the functions $P_l(\cos\theta)$ are the Legendre polynomials defined in Section 2.5.

Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be two vectors having polar angles (θ_1, ϕ_1) and (θ_2, ϕ_2) respectively, and let θ be the angle between them. It can be shown that

$$P_l(\cos\theta) = \frac{4\pi}{2l + 1} \sum_{m=-l}^{+l} Y^{\star}_{lm}(\theta_1, \phi_1) Y_{lm}(\theta_2, \phi_2) \qquad [A4.23]$$

which is known as the addition (or biaxial) theorem of the spherical harmonics. From the generating function [2.168] of the Legendre polynomials, we see that

$$\frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|} = \sum_{l=0}^{\infty} \frac{(r_<)^l}{(r_>)^{l+1}} P_l(\cos\theta) \qquad [A4.24]$$

which, using [A4.23], may also be written as

$$\frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|} = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} \frac{4\pi}{2l + 1} \frac{(r_<)^l}{(r_>)^{l+1}} Y^{\star}_{lm}(\theta_1, \phi_1) Y_{lm}(\theta_2, \phi_2) \qquad [A4.25]$$

It can also be shown (Mathews and Walker, 1973) that

$$\frac{\exp[ik|\mathbf{r}_1 - \mathbf{r}_2|]}{|\mathbf{r}_1 - \mathbf{r}_2|} = ik \sum_{l=0}^{\infty} (2l + 1) j_l(kr_<)[j_l(kr_>) + in_l(kr_>)] P_l(\cos\theta) \qquad [A4.26]$$

where j_l and n_l are spherical Bessel and Neumann functions respectively. Finally, we quote the formula giving the expansion of a plane wave in Legendre polynomials, namely

$$e^{i\mathbf{k}\cdot\mathbf{r}} = \sum_{l=0}^{\infty} (2l + 1) i^l j_l(kr) P_l(\cos\theta) \qquad [A4.27]$$

where θ is the angle between the vectors $\mathbf{k}$ and $\mathbf{r}$.

Addition of angular momenta. The Clebsch–Gordan coefficients

Consider a system described by two angular momenta $\mathbf{J}_1$ and $\mathbf{J}_2$, such that the components of $\mathbf{J}_1$ commute with the components of $\mathbf{J}_2$. For example, $\mathbf{J}_1$ and $\mathbf{J}_2$ could be the angular momenta of different particles, or the orbital and spin angular momenta of a single particle. The normalised simultaneous eigenfunctions of $\mathbf{J}_1^2$ and J_{1z} corresponding to eigenvalues $j_1(j_1 + 1)\hbar^2$ and $m_1\hbar$ will be denoted by $\psi_{j_1m_1}$ and similarly, the normalised eigenfunctions of $\mathbf{J}_2^2$ and J_{2z} corresponding to eigenvalues $j_2(j_2 + 1)\hbar^2$ and $m_2\hbar$ will be denoted by $\psi_{j_2m_2}$. The simultaneous eigenfunctions of $\mathbf{J}_1^2, J_{1z}, \mathbf{J}_2^2$ and J_{2z} are then given by the product functions

$$\psi_{j_1m_1;j_2m_2} = \psi_{j_1m_1} \times \psi_{j_2m_2} \qquad \text{[A4.28]}$$

and for a given j_1 and j_2, there are $(2j_1 + 1) \times (2j_2 + 1)$ of these functions.

Now consider the total angular momentum

$$\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2 \qquad \text{[A4.29]}$$

Since $\mathbf{J}^2, \mathcal{J}_z, \mathbf{J}_1^2$ and $\mathbf{J}_2^2$ all commute, these operators possess a set of simultaneous eigenfunctions, which we shall write as $\Phi^{jm}_{j_1j_2}$ where

$$\begin{aligned} \mathbf{J}^2\Phi^{jm}_{j_1j_2} &= j(j+1)\hbar^2\Phi^{jm}_{j_1j_2} \\ \mathcal{J}_z\Phi^{jm}_{j_1j_2} &= m\hbar\Phi^{jm}_{j_1j_2} \end{aligned} \qquad \text{[A4.30]}$$

For a given j, there are $(2j + 1)$ values of m with $-j \leqslant m \leqslant j$ and j can take any of the values $|j_1 - j_2|, |j_1 - j_2| + 1, \ldots (j_1 + j_2)$. Again there are $(2j_1 + 1) \times (2j_2 + 1)$ of the functions $\Phi^{jm}_{j_1j_2}$, which can be related to the function [A4.28] by a unitary transformation:

$$\Phi^{jm}_{j_1j_2} = \sum_{m_1m_2} \langle j_1j_2m_1m_2|jm\rangle\psi_{j_1m_1;\, j_2m_2} \qquad \text{[A4.31]}$$

The coefficients $\langle j_1j_2m_1m_2|jm\rangle$ are called Clebsch–Gordan coefficients. These coefficients vanish unless $m = m_1 + m_2$ and $|j_1 - j_2| \leqslant j \leqslant j_1 + j_2$, and possess the following important properties:

Orthonormality relations

$$\sum_{m_1m_2} \langle j_1j_2m_1m_2|jm\rangle\langle j_1j_2m_1m_2|j'm'\rangle = \delta_{jj'}\,\delta_{mm'}$$

$$\sum_{j,m} \langle j_1j_2m_1m_2|jm\rangle\langle j_1j_2m_1'm_2'|jm\rangle = \delta_{m_1m_1'}\delta_{m_2m_2'} \qquad \text{[A4.32]}$$

Symmetry properties

$$\begin{aligned} \langle j_1j_2m_1m_2|jm\rangle &= (-1)^{j_1+j_2-j}\,\langle j_2j_1m_2m_1|jm\rangle \\ &= (-1)^{j_1+j_2-j}\,\langle j_1j_2 - m_1 - m_2|j - m\rangle \\ &= (-1)^{j_1-m_1}\left(\frac{2j+1}{2j_2+1}\right)^{1/2}\langle j_1jm_1 - m|j_2 - m_2\rangle \end{aligned} \qquad \text{[A4.33]}$$

In Table A4.1, the coefficients $\langle j_1j_2m_1m_2|jm\rangle$ are tabulated for the cases $j_2 = \frac{1}{2}$ and $j_2 = 1$. By using the symmetry relations, all the coefficients with any one of j_1, j_2 or j equal to $\frac{1}{2}$, or to 1, can be found.

Useful notations

When adding two orbital angular momenta $\mathbf{L}_1$ and $\mathbf{L}_2$, we shall write [A4.31] in the explicit position representation as

$$\mathcal{Y}^{lm}_{l_1l_2}(\theta_1\phi_1;\, \theta_2\phi_2) = \sum_{m_1m_2} \langle l_1l_2m_1m_2|lm\rangle\, Y_{l_1m_1}(\theta_1, \phi_1)Y_{l_2m_2}(\theta_2, \phi_2) \qquad \text{[A4.34]}$$

Table A4.1 Clebsch–Gordan coefficients for $j_2 = \frac{1}{2}$ and $j_2 = 1$

$\langle j_1 \tfrac{1}{2} m_1 m_2 | jm\rangle$

j	$m_2 = \frac{1}{2}$	$m_2 = -\frac{1}{2}$
$j_1 + \frac{1}{2}$	$\left(\dfrac{j_1 + m + \frac{1}{2}}{2j_1 + 1}\right)^{1/2}$	$\left(\dfrac{j_1 - m + \frac{1}{2}}{2j_1 + 1}\right)^{1/2}$
$j_1 - \frac{1}{2}$	$-\left(\dfrac{j_1 - m + \frac{1}{2}}{2j_1 + 1}\right)^{1/2}$	$\left(\dfrac{j_1 + m + \frac{1}{2}}{2j_1 + 1}\right)^{1/2}$

$\langle j_1 1 m_1 m_2 | jm\rangle$

j	$m_2 = 1$	$m_2 = 0$	$m_2 = -1$
$j_1 + 1$	$\left[\dfrac{(j_1 + m)(j_1 + m + 1)}{(2j_1 + 1)(2j_1 + 2)}\right]^{1/2}$	$\left[\dfrac{(j_1 - m + 1)(j_1 + m + 1)}{(2j_1 + 1)(j_1 + 1)}\right]^{1/2}$	$\left[\dfrac{(j_1 - m)(j_1 - m + 1)}{(2j_1 + 1)(2j_1 + 2)}\right]^{1/2}$
j_1	$-\left[\dfrac{(j_1 + m)(j_1 - m + 1)}{2j_1(j_1 + 1)}\right]^{1/2}$	$\left[\dfrac{m^2}{j_1(j_1 + 1)}\right]^{1/2}$	$\left[\dfrac{(j_1 - m)(j_1 + m + 1)}{2j_1(j_1 + 1)}\right]^{1/2}$
$j_1 - 1$	$\left[\dfrac{(j_1 - m)(j_1 - m + 1)}{2j_1(2j_1 + 1)}\right]^{1/2}$	$-\left[\dfrac{(j_1 - m)(j_1 + m)}{j_1(2j_1 + 1)}\right]^{1/2}$	$\left[\dfrac{(j_1 + m + 1)(j_1 + m)}{2j_1(2j_1 + 1)}\right]^{1/2}$

where $\mathcal{Y}^{lm}_{l_1 l_2}$ is a simultaneous eigenfunction of $\mathbf{L}_1^2$, $\mathbf{L}_2^2$, $\mathbf{L}^2$ and L_z, and $\mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2$. Similarly, when adding an orbital angular momentum $\mathbf{L}$ with a spin angular momentum $\mathbf{S}$, so that $\mathbf{J} = \mathbf{L} + \mathbf{S}$ we shall often write

$$\mathcal{Y}^{jm}_{ls}(\theta, \phi) = \sum_{m_l m_s} \langle l s m_l m_s | jm\rangle Y_{lm_l}(\theta, \phi)\, \chi_{sm_s} \qquad \text{[A4.35]}$$

where χ_{sm_s} is a spin wave function.

When taking matrix elements of operators with respect to the eigenfunctions $\Phi^{jm}_{j_1 j_2}$, $\mathcal{Y}^{jm}_{ls}$ . . . , we shall frequently use the Dirac notation in which eigenvectors are written in the form $\Phi^{jm}_{j_1 j_2} \to |j_1 j_2 jm\rangle$. We then write

$$\langle \Phi^{jm}_{j_1 j_2} | A | \Phi^{j'm'}_{j'_1 j'_2}\rangle = \langle j_1 j_2 jm | A | j'_1 j'_2 j' m'\rangle \qquad \text{[A4.36]}$$

and

$$\int d\Omega_1 \int d\Omega_2\, \mathcal{Y}^{lm*}_{l_1 l_2}(\theta_1\phi_1;\, \theta_2\phi_2) A \mathcal{Y}^{l'm'}_{l'_1 l'_2}(\theta_1\phi_1;\, \theta_2\phi_2) = \langle l_1 l_2 lm | A | l'_1 l'_2 l' m'\rangle \qquad \text{[A4.37]}$$

where A is an operator. The defining relation of the Clebsch–Gordan coefficients [A4.31] can then be written as

$$|j_1 j_2 jm\rangle = \sum_{m_1 m_2} \langle j_1 j_2 m_1 m_2 | jm\rangle |j_1 m_1\rangle \times |j_2 m_2\rangle \qquad \text{[A4.38]}$$

Integrals of products of spherical harmonics

It can be shown that the product $Y_{l_1m_1}(\theta, \phi)Y_{l_2m_2}(\theta, \phi)$ can be expressed as a series by

$$Y_{l_1m_1}(\theta, \phi)Y_{l_2m_2}(\theta, \phi) = \sum_{l=|l_1-l_2|}^{l_1+l_2} \sum_{m=-l}^{l} \left[\frac{(2l_1 + 1)(2l_2 + 1)}{4\pi(2l + 1)}\right]^{1/2} \times \langle l_1 l_2 00|l0\rangle\langle l_1 l_2 m_1 m_2|lm\rangle Y_{lm}(\theta, \phi) \qquad \text{[A4.39]}$$

This enables us to evaluate the integral of a product of three spherical harmonics. Using [2.181b] and the orthonormality property [A4.18] we have

$$\int Y_{l_1m_1}(\theta, \phi)Y_{l_2m_2}(\theta, \phi)Y_{l_3m_3}(\theta, \phi)\, d\Omega = (-1)^{m_3}\left[\frac{(2l_1 + 1)\,(2l_2 + 1)}{4\pi(2l_3 + 1)}\right]^{1/2}\langle l_1 l_2 00|l_3 0\rangle\langle l_1 l_2 m_1 m_2|l_3 - m_3\rangle \qquad \text{[A4.40]}$$

Scalar and vector operators

A scalar operator $\mathcal{S}$ is one for which the expectation values $\langle\phi|\mathcal{S}|\phi\rangle$ are unaltered by a rotation of the coordinate system. It can be shown (Powell and Crasemann, 1962; Merzbacher, 1970) that for an operator to be scalar it must commute with all components of the total angular momentum operator $\mathbf{J}$:

$$[\mathcal{S}, \mathbf{J}] = 0 \qquad \text{[A4.41]}$$

from which it follows that if ψ_{jm} is a simultaneous eigenfunction of $\mathbf{J}^2$ and $\mathcal{J}_z$ belonging to the quantum numbers j and m, then $\langle\psi_{jm}|\mathcal{S}|\psi_{j'm'}\rangle$ vanishes unless $j = j'$ and $m = m'$, and

$$\mathcal{S}\psi_{jm} = \lambda\psi_{jm} \qquad \text{[A4.42]}$$

where λ is an eigenvalue of $\mathcal{S}$. Since $\mathcal{J}_+\psi_{jm} = N\psi_{jm+1}$ and as $[\mathcal{J}_+, \mathcal{S}] = 0$, we must also have

$$\mathcal{S}\psi_{jm+1} = \lambda\psi_{jm+1} \qquad \text{[A4.43]}$$

so the eigenvalue λ is independent of m (but it does depend on j).

In general, wave functions depend on other quantum numbers in addition to angular momentum quantum numbers (for example, principal quantum numbers). Denoting these other quantum numbers collectively by α, we see that

$$\langle\phi_{\alpha jm}|\mathcal{S}|\phi_{\alpha'j'm'}\rangle \equiv \langle\alpha jm|\mathcal{S}|\alpha'j'm'\rangle = \lambda_{j\alpha\alpha'}\delta_{jj'}\delta_{mm'} \qquad \text{[A4.44]}$$

This is the simplest example of the fact that matrix elements of operators having well-defined properties under rotations depend upon the magnetic quantum numbers through a 'geometrical' factor (equal in the present case to $\delta_{mm'}$) which is independent of the dynamics of the system.

In this book, we are mainly concerned with *vector operators*, of which the components transform like the components of a vector under rotations. The condition for an operator **V** with components V_x, V_y, V_z to be a vector operator is that it satisfies the commutation relations [5.71]. It is useful to define the spherical components of **V** as

$$V_1 = -\frac{1}{\sqrt{2}}(V_x + iV_y); \qquad V_0 = V_z; \qquad V_{-1} = \frac{1}{\sqrt{2}}(V_x - iV_y) \quad \text{[A4.45]}$$

The set of three operators V_q ($q = -1, 0, +1$) satisfies the commutation relations (which follow from [5.71])

$$\begin{aligned} [J_z, V_q] &= q\hbar V_q \\ [J_+, V_q] &= [(1-q)(2+q)]^{1/2}\hbar V_{q+1} \\ [J_-, V_q] &= [(1+q)(2-q)]^{1/2}\hbar V_{q-1} \end{aligned} \quad \text{[A4.46]}$$

where $V_q = 0$ if $q \neq 0, \pm 1$. The operators V_q are special cases of *irreducible tensor operators* T_q^k of rank k, which form a set of $(2k + 1)$ operators with q running over the values $-k, -k+1, \ldots k-1, k$, and satisfying the commutation relations

$$\begin{aligned} [J_z, T_q^k] &= q\hbar T_q^k \\ [J_+, T_q^k] &= [(k-q)(k+q+1)]^{1/2}\hbar T_{q+1}^k \\ [J_-, T_q^k] &= [(k+q)(k-q+1)]^{1/2}\hbar T_{q-1}^k \end{aligned} \quad \text{[A4.47]}$$

For such operators, the matrix elements between eigenfunctions ψ_{jm} and $\psi_{j'm'}$ depend on m and m' only through a factor which can be shown to be equal to the Clebsch–Gordan coefficient $\langle jkmq|j'm'\rangle$. Thus

$$\langle \alpha' j' m' | T_q^k | \alpha j m\rangle = \frac{1}{\sqrt{2j'+1}}\langle jkmq|j'm'\rangle\langle \alpha' j' \| T^k \| \alpha j\rangle \quad \text{[A4.48]}$$

where the reduced matrix element $\langle \alpha' j' \| T^k \| \alpha j\rangle$ is a number depending on $\alpha\alpha' jj'$ but not on m and m'. This result is called the *Wigner–Eckart theorem*. It should be noticed that the appearance of the factor $(2j' + 1)^{-1/2}$ on the right-hand side of [A4.48] is conventional; it could be absorbed into the reduced matrix element.

The application of the Wigner–Eckart theorem to a scalar operator $\mathcal{S} = T_0^0$ reproduces the result [A4.44]. For vector operators we have $V_q \equiv T_q^1$. As an example of a vector operator we can take the total angular momentum, **J**. Then definining J_q by using [A4.45] with J in place of V we have,

$$\langle \alpha' j' m' | J_q | \alpha j m\rangle = \frac{1}{\sqrt{2j'+1}}\langle j1mq|j'm'\rangle\langle \alpha' j' \| J_q \| \alpha j\rangle \quad \text{[A4.49]}$$

Setting $q = 0$ and noting that $J_0 \equiv J_z$, we find by using Table A4.1 that

$$\langle \alpha' j' \| J \| \alpha j\rangle = \sqrt{j(j+1)}\hbar\, \delta_{jj'}\, \delta_{\alpha\alpha'} \quad \text{[A4.50]}$$

It follows from [A4.48] that if **V** is any vector operator then

$$\langle \alpha' j' m' | \mathbf{V} | \alpha j m \rangle = C \langle \alpha' j' m' | \mathbf{J} | \alpha j m \rangle \qquad [A4.51]$$

where C is independent of m and m'. For the case $j' = j$, $\alpha' = \alpha$, C can be found by writing

$$\langle \alpha j m | \mathbf{V} \cdot \mathbf{J} | \alpha j m \rangle = \sum_{m'} \langle \alpha j m | \mathbf{V} | \alpha j m' \rangle \cdot \langle \alpha j m' | \mathbf{J} | \alpha j m \rangle \qquad [A4.52]$$

where we have used the closure property

$$\sum_{m'} | \alpha j m' \rangle \langle \alpha j m' | = 1 \qquad [A4.53]$$

Thus

$$\begin{aligned} \langle \alpha j m | \mathbf{V} \cdot \mathbf{J} | \alpha j m \rangle &= C \sum_{m'} \langle \alpha j m | \mathbf{J} | \alpha j m' \rangle \cdot \langle \alpha j m' | \mathbf{J} | \alpha j m \rangle \\ &= C \langle \alpha j m | \mathcal{J}^2 | \alpha j m \rangle \\ &= C j(j+1)\hbar^2 \end{aligned} \qquad [A4.54]$$

Using [A4.51], we then obtain the useful equation

$$\begin{aligned} j(j+1)\hbar^2 \langle \alpha j m' | \mathbf{V} | \alpha j m \rangle &= \langle \alpha j m | \mathbf{V} \cdot \mathbf{J} | \alpha j m \rangle \langle \alpha j m' | \mathbf{J} | \alpha j m \rangle \\ &= \langle \alpha j m' | (\mathbf{V} \cdot \mathbf{J}) \mathbf{J} | \alpha j m \rangle \end{aligned} \qquad [A4.55]$$

which relates the expectation values of the components of **V** to those of the components of **J**.