Many-Body Problem with Strong Forces

ROBERT JASTROW*†
Nucleonics Division, Naval Research Laboratory, Washington, D. C., and Sloane Physics Laboratory, Yale University, New Haven, Connecticut
(Received January 26, 1955)

A trial solution constructed from two-particle functions is applied to the problem of the N-particle system with strong interactions. In a variational treatment based on this wave function the expectation values, which cannot be factored into single-particle integrals, are evaluated by a cluster development in powers of the particle density. The procedure is illustrated by a calculation of the pair distribution function and zero-point energy of the hard sphere gas, for Bose and Fermi statistics.

IN the present paper we examine, by a variational method, the ground state properties of a quantum mechanical system of N particles of equal mass m, contained in a volume Ω, and interacting through central two-body forces represented by the potential $V(r)$. Our interest is primarily in potentials $V(r)$ of short range relative to the mean particle spacing.

It will be assumed that both N and Ω are large enough to permit the neglect of surface effects; the properties of the ground state then depend only on m, $V(r)$, and the particle density, $n=N/\Omega$.

The Hamiltonian of the system is

$$H=-\frac{\hbar^2}{2m}\sum_{i=1}^{N}\nabla_i^2+\sum_{i<j=1}^{N}V(r_{ij}). \tag{1}$$

The eigenfunctions ψ of (1) are usually approximated by (symmetrized or antisymmetrized) products of one-particle functions,

$$\psi\sim\prod_{i=1}^{N}\varphi_i(r_i), \tag{2}$$

the effect of the interactions $V(r_{ij})$ being introduced as a perturbation on the independent particle motions; an upper bound on the energy of the ground state is then obtained by computing[1] $\bar{H}=\int\psi^*H\psi/\int\psi^*\psi$ from (2) and minimizing with respect to variations in the functions φ_i. This procedure fails in problems involving strong interactions, however, through the omission of correlations in particle position from (2). As an example we consider the repulsive potential,

$$\begin{aligned} &V=|V_0|, \quad r_{ij}\leq r_0;\\ &V=0, \qquad r_{ij}>r_0. \end{aligned} \tag{3}$$

The mean potential energy per particle, $\bar{V}/N$, computed from (2) and (3) is seen to be of the order of $(Nr_0^3/\Omega)|V_0|$; if now (3) goes into the hard sphere interaction ($|V_0|\to\infty$), the upper bound provided by $\bar{V}$ increases without limit.

The estimate of $\bar{V}$ is improved if (2) is replaced by a form containing the correlations implied by the relations,

$$\psi(r_1\cdots r_N)\approx 0, \quad r_{ij}<r_0, \tag{4}$$

which the eigenfunctions of (1) must satisfy when $V(r)$ is a strong repulsion. The conditions (4) are easily fulfilled by trial functions which depend explicitly on the interparticle separations r_{ij}; of such functions, the simplest is the product over all $N(N-1)/2$ pairs,[2]

$$\psi\sim\prod_{i<j=1}^{N}f(r_{ij}), \tag{5}$$

with $f(r_{ij})$ defined to vanish for $r<r_0$ and to approach unity for $r\gg r_0$.[3]

The particular example of the hard sphere gas requires for $f(r)$ a form such as

$$\begin{aligned} &f=0, && r\leq r_0;\\ &f=1-\frac{e^{-\beta(r-r_0)}}{r/r_0}, && r>r_0, \end{aligned} \tag{6}$$

in which β is an arbitrary parameter. For the variational application of (5) and (6),

$$\bar{H}=\int\prod_{i<j=1}^{N}f^*(r_{ij})H\prod_{i<j=1}^{N}f(r_{ij})\Big/\int\prod_{i<j=1}^{N}|f(r_{ij})|^2 \tag{7}$$

must be evaluated and minimized with respect to variations in β.

The form (5) may be applied to other interactions $V(r_{ij})$ if $f(r_{ij})$ is chosen more generally as an approximation to the eigenfunctions of the two-body Hamiltonian

$$-\frac{\hbar^2}{2m}(\nabla_i^2+\nabla_j^2)+V(r_{ij}) \tag{8}$$

* Present address: Nucleonics Division, Naval Research Laboratory, Washington, D. C.

† Part of this work was assisted by the joint program of the Office of Naval Research and the U. S. Atomic Energy Commission and by the Office of Ordnance Research, U. S. Army.

[1] Integrals are taken over the $3N$ coordinates of all particles unless otherwise indicated.

[2] This form has already been suggested by N. F. Mott for the hard sphere Bose gas, and applied in the low-density limit by R. B. Dingle [Phil. Mag. **40**, 573 (1949)].

[3] As $r\to\infty$, $f(r)$ must approach a constant amplitude, conveniently chosen as unity. If instead $f(r)$ were to vanish for large r, ψ would vanish almost everywhere in the configuration space of the system.

within the range of V; as before, f must approach unity at large distances.

Although (5) is constructed from two-body functions and apparently includes only pair correlations, it should be noted that effects involving ≥ 3 particles are also present: for example, the 3-particle group (i,j,k) is connected in (6) by the factor $[f(r_{ij})f(r_{jk})f(r_{ki})]$, and particles i, j, k, and l by factors such as $[f(r_{ij})f(r_{jk}) \times f(r_{kl})f(r_{li})]$. The contributions of these terms appear explicitly in the cluster development of $\bar{H}$ to be given below.

CLUSTER DEVELOPMENT $\bar{H}$

The expectation value of H may be computed from (7):

$$\bar{H}=\frac{\int \prod_{i<j=1}^{N} f^*(r_{ij})\left[-\frac{\hbar^2}{2m}N\nabla_1^2+\tfrac{1}{2}N(N-1)V(r_{12})\right]\prod_{i<j=1}^{N} f(r_{ij})}{\int \prod_{i<j=1}^{N} |f(r_{ij})|^2}.$$

The cross terms in the kinetic energy vanish in the integration and we obtain

$$H=N(N-1)\frac{\int \prod_{i<j=1}^{N} f^*(r_{ij})\left[-\frac{\hbar^2}{2m}\frac{\nabla_1^2 f(r_{12})}{f(r_{12})}+\tfrac{1}{2}V(r_{12})\right]\prod_{i<j=1}^{N} f(r_{ij})}{\int \prod_{i<j=1}^{N} |f(r_{ij})|^2}. \tag{9}$$

Because f approaches unity outside the range of forces, (9) is identical in form with the configuration integrals encountered in the classical theory of the imperfect gas.[4] We treat it similarly, writing for the energy per particle

$$\frac{1}{N}\bar{H}=\frac{N-1}{\Omega}\int\left[-\frac{\hbar^2}{2m}\frac{\nabla_1^2 f(r_{12})}{f(r_{12})}+\tfrac{1}{2}V(r_{12})\right] \times g(r_{12})d^3r_{12}. \tag{10}$$

The pair distribution function, $g(r_{12})$, is defined by integration of ψ^2 over particles $3, \cdots, N$:

$$g(r_{12})=\Omega^2\frac{\int \prod_{i<j=1}^{N} |f(r_{ij})|^2 d^3r_3\cdots d^3r_N}{\int \prod_{i<j=1}^{N} |f(r_{ij})|^2 d^3r_1\cdots d^3r_N};$$

it determines the probability of a separation r_{12} between particles 1 and 2.

The function $g(r)$ may be expanded in ascending powers of particle density $n=N/\Omega$, following the cluster development of Ursell and Mayer.[5] Defining

$$h(r_{ij})=f^2(r_{ij})-1,$$

we have

$$g(r_{12})=f^2(r_{12})[g^{(2)}(r_{12})+ng^{(3)}(r_{12}) +n^2g^{(4)}(r_{12})+\cdots], \tag{11}$$

where

$$g^{(2)}(r_{12})=1, \tag{11a}$$

$$g^{(3)}(r_{12})=\int h(r_{13})h(r_{32})d^3r_3, \tag{11b}$$

$$g^{(4)}(r_{12})=\tfrac{1}{2}[g^{(3)}(r)]^2+\int[2h(r_{14})h(r_{43})h(r_{32}) +4h(r_{13})h(r_{32})h(r_{24})h(r_{43}) +h(r_{41})h(r_{13})h(r_{32})h(r_{24})h(r_{43})]d^3r_3d^3r_4. \tag{11c}$$

The leading term in (11) describes the correlation which results from the explicit dependence of ψ on r_{12} through the factor $f(r_{12})$. The next term ($g^{(3)}$) is associated with the distortion of ψ in its dependence on r_{12}, resulting from the presence of a third particle in the neighborhood of 1 and 2, etc.

Bose Particles

The function (5) is symmetrical in all particles and directly applicable to Einstein-Bose systems. We have for the expectation values of the potential and kinetic energies per particle,

$$\frac{1}{N}\bar{V}=4\pi n\int r^2V(r)g(r)dr, \tag{12}$$

and

$$\frac{1}{N}\bar{T}=4\pi n\int r^2\left[\frac{f''+(2/r)f'}{f}\right]g(r)dr, \tag{13}$$

with $g(r)$ given by (11).

[4] $e^{-V(r_{ij})/kT}$ replaces $f^2(r_{ij})$ in the classical analog.

[5] The cluster development is reviewed by J. De Boer in Repts. Progr. Phys. **12**, 305 (1948).

Fermi Particles

A modification of (5) appropriate to antisymmetrical systems is

$$\psi\sim S(\mathbf{r}_1\cdots\mathbf{r}_N)\prod_{i<j=1}^{N} f(r_{ij}), \tag{14}$$

where S is a Slater determinant of one-particle functions, $\varphi_l(\mathbf{r}_m)$. We choose plane waves for these: $\varphi_l(\mathbf{r}_m)=e^{i\mathbf{k}l\cdot\mathbf{r}_m}$.

The corresponding modifications in the cluster development of $\bar{H}/N$ are readily obtained. The potential energy again takes the form

$$\frac{1}{N}\bar{V}=4\pi n\int r^2V(r)g(r)dr, \tag{15}$$

with

$$g(r)=f^2(r)[g^{(2)}(r)+ng^{(3)}(r)+\cdots], \tag{16}$$

where now

$$g^{(2)}(r_{12})=\sum_{\alpha_1\alpha_2=1}^{N}\exp[-i(\mathbf{k}_{\alpha_1}\cdot\mathbf{r}_1+\mathbf{k}_{\alpha_2}\cdot\mathbf{r}_2)]\times\begin{vmatrix}\exp(i\mathbf{k}_{\alpha_1}\cdot\mathbf{r}_1) & \exp(i\mathbf{k}_{\alpha_2}\cdot\mathbf{r}_1)\\ \exp(i\mathbf{k}_{\alpha_1}\cdot\mathbf{r}_2) & \exp(i\mathbf{k}_{\alpha_2}\cdot\mathbf{r}_2)\end{vmatrix}, \tag{17}$$

$$g^{(3)}(r_{12})=\int d^3r_3h(r_{13})h(r_{32})\sum_{\alpha_1\alpha_2\alpha_3=1}^{N}\exp[-i(\mathbf{k}_{\alpha_1}\cdot\mathbf{r}_1+\mathbf{k}_{\alpha_2}\cdot\mathbf{r}_2+\mathbf{k}_{\alpha_3}\cdot\mathbf{r}_3)]\times\begin{vmatrix}\exp(ik_{\alpha_1}\cdot r_1) & \exp(ik_{\alpha_2}\cdot r_1) & \exp(ik_{\alpha_3}\cdot r_1)\\ \exp(ik_{\alpha_1}\cdot r_2) & \exp(ik_{\alpha_2}\cdot r_2) & \exp(ik_{\alpha_3}\cdot r_2)\\ \exp(ik_{\alpha_1}\cdot r_3) & \exp(ik_{\alpha_2}\cdot r_3) & \exp(ik_{\alpha_3}\cdot r_3)\end{vmatrix}. \tag{18}$$

The sums in (17) and (18) are taken over all states within the Fermi sphere of radius $k_F=(6\pi^2n)^{\frac{1}{3}}$. We find

$$g^{(2)}(r_{12})=1-l^2(k_Fr_{12}), \tag{17'}$$

$$g^{(3)}(r_{12})=\int d^3r_3h(r_{13})h(r_{32})\times\{1-2[l^2(k_Fr_{12})+l^2(k_Fr_{13})+l^2(k_Fr_{23})]+l(k_Fr_{12})l(k_Fr_{13})l(k_Fr_{23})\}, \tag{18'}$$

with

$$l(x)=3(\sin x-x\cos x)/x^3. \tag{19}$$

The kinetic energy operator produces three terms:

$$\frac{1}{N}\bar{T}=-\frac{\hbar^2}{2m}\langle S\nabla^2(\prod f)\rangle_{Av}+\langle(\prod f)\nabla^2S\rangle_{Av}+\langle 2\nabla S\cdot\nabla(\prod f)\rangle_{Av}. \tag{20}$$

These terms, in order of their appearance in (20), are identified as the correlational kinetic energy ($\bar{T}_C$), the Fermi energy ($\bar{T}_F$), and the cross term ($\bar{T}_{CF}$).

Application of the cluster development to (20) yields the following results:

(a) Correlational kinetic energy:

$$\frac{1}{N}\bar{T}_C=-\frac{\hbar^2}{2m}n\int\left[\frac{f''+(2/r)f'}{f}\right]g(r)d^3r, \tag{21}$$

with $g(r)$ defined by (16)–(18);

(b) Fermi energy:

$$\frac{1}{N}\bar{T}_F=\frac{\hbar^2}{2m}\frac{1}{N}\sum_{\alpha_1=1}^{N}k_{\alpha_1}{}^2\times\left[1+n\left(\frac{1}{N^2}\sum_{\alpha_2\alpha_3=1}^{N}(Q_{\alpha_1\alpha_2}-Q_{\alpha_2\alpha_3}{}^*)\right)+\cdots\right], \tag{22}$$

where

$$Q_{\alpha_i\alpha_j}=\int h(r_{ij})\times\exp[-i(\mathbf{k}_{\alpha_i}\cdot\mathbf{r}_1+\mathbf{k}_{\alpha_j}\cdot\mathbf{r}_2)]D^{(2)}(ij;12)\frac{d^3r_2}{\Omega}. \tag{23}$$

(c) Cross term:

$$\frac{1}{N}\bar{T}_{CF}=-2i\frac{\hbar^2}{2m}n\int d^3r_{12}f(r_{12})\nabla f(r_{12})\cdot\left\{\frac{1}{N}\sum_{\alpha_1=1}^{N}\mathbf{k}_{\alpha_1}\times\left[\frac{1}{N}\sum_{\alpha_2=1}^{N}\exp[-i(\mathbf{k}_{\alpha_1}\cdot\mathbf{r}_1+\mathbf{k}_{\alpha_2}\cdot\mathbf{r}_2)]D^{(2)}(12;12)+n\int d^3r_3h(r_{13})h(r_{23})\frac{1}{N^2}\sum_{\alpha_2\alpha_3=1}^{N}\exp[-i(\mathbf{k}_{\alpha_1}\cdot\mathbf{r}_1+\mathbf{k}_{\alpha_2}\cdot\mathbf{r}_2+\mathbf{k}_{\alpha_3}\cdot\mathbf{r}_3)]\times D^{(3)}(123;123)\right]\right\}. \tag{24}$$

$D^{(2)}$ and $D^{(3)}$ denote the determinants occurring in (17) and (18), respectively.

Expansion of the Energy in Powers of Particle Density

The expansion of $\bar{H}$ in integral powers of n is directly given, for Bose particles, by insertion of (11) in (12) and (13).

The corresponding development of $\bar{H}$ for Fermi particles is complicated by the density dependence of the Fermi wave number, $k_F=(6\pi^2n)^{\frac{1}{3}}$, occurring in (17)–(24). A development in mixed powers of n and $n^{\frac{1}{3}}$ is obtained by first expanding (18), (22), and (23) in powers of $(k_Fr_{ij})^2$. The results are given in the Appendix.

HARD-SPHERE GAS

We apply these formulas to a calculation of the kinetic energy of a gas of hard spheres, with interaction radius r_0, for Bose and Fermi statistics. It will be seen

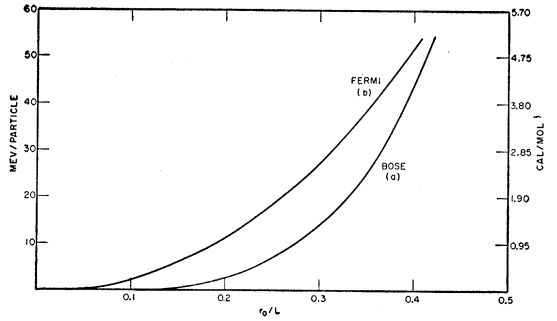

FIG. 1. The kinetic energy of a hard-sphere gas with (a) Bose and (b) Fermi statistics: The abscissae represent the ratio of hard sphere interaction radius (r_0) to mean particle spacing (L); the ordinates on the left refer to particles with nucleon mass and $r_0=10^{-13}$ cm, those on the right to particles with the mass of He^4 and $r_0=2.5\times10^{-8}$ cm.

that the results take the form,

$$\bar{T}=(\hbar^2/2mr_0^2)t, \tag{25}$$

in which t depends only on βr_0 and r_0/L, where $L=n^{-\frac{1}{3}}$ indicates the mean particle spacing.

The calculation is first made for a Bose gas, to lowest order in n, for several choices on the form of $f(r)$:

$$\left.\begin{aligned}&\text{(i)}\quad \text{Gaussian: } f=1-\exp[-\beta^2(r-r_0)^2]\\ &\text{(ii)}\quad \text{Exponential: } f=1-e^{-\beta(r-r_0)}\\ &\text{(iii)}\quad \text{Yukawa: } f=1-\frac{e^{-\beta(r-r_0)}}{r/r_0}\end{aligned}\right\} r\geq r_0,$$

$$f=0, \qquad r<r_0.$$

The leading term in the density expansion of the kinetic energy is

$$\frac{1}{N}\bar{T}=4\pi n\int f\left(f''+\frac{2}{r}f'\right)r^2dr. \tag{26}$$

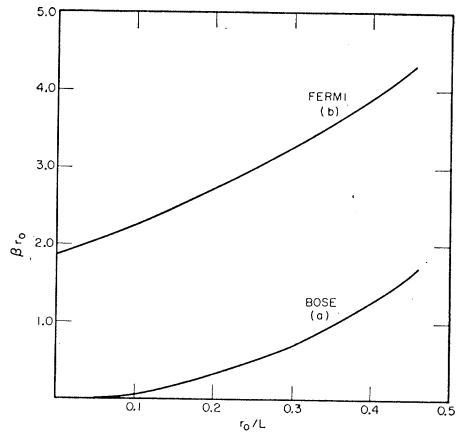

FIG. 2. Values of the variational parameter (βr_0) at the minimum in $\bar{T}$, for a hard-sphere gas with the indicated statistics.

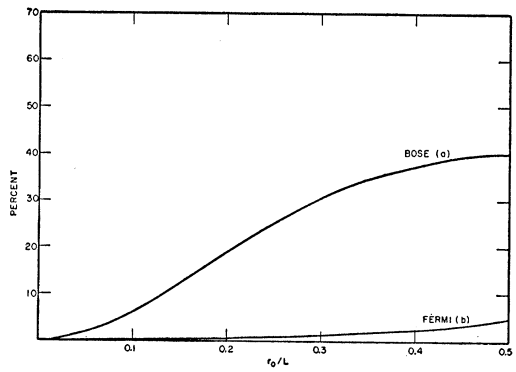

FIG. 3. The fraction contributed to the kinetic energy of a hard-sphere gas by the three-body term ($g^{(3)}$) in the cluster development.

Evaluation of (26) for the several forms of f and minimization with respect to β leads to the following results:

$$\frac{1}{N}\bar{T}=B\frac{\hbar^2}{2mr_0^2}\left(\frac{r_0}{L}\right)^3. \tag{27}$$

The constant B and the value of β at the minimum in $\bar{T}$ are given in Table I.

The Yukawa function, which varies as $1-r_0/r$ near the sphere boundary in agreement with the exact solution to (8), yields the lowest kinetic energy, and will be assumed for $f(r)$ in the remainder of our calculations.

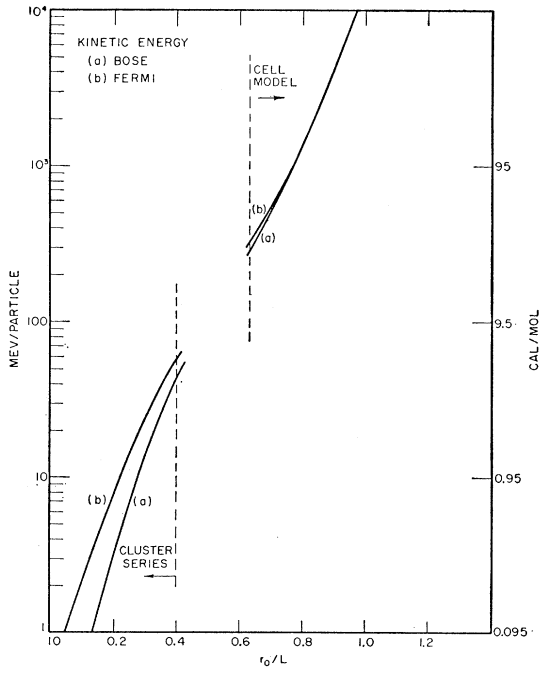

FIG. 4. The kinetic energy of a hard-sphere gas; the low-density values are obtained from Fig. 1; those at high density are estimated from a cell model. The ordinates are as in Fig. 1.

The next (n^2) term in the cluster development introduces a three-body integral which may be reduced to combinations of exponential integrals. The variation of $\bar{T}$ must now be carried out numerically; the results of the computation are shown in Fig. 1, with terms included to order n^2 and $n^{7/3}$, respectively, for Bose and Fermi gases. The ordinate scale on the left refers to particles of nucleon mass and to a hard sphere interaction radius of 1.0×10^{-13} cm, that on the right to particles with the mass of He^4 and an interaction radius of 2.5×10^{-8} cm. Values of $\bar{T}$ for other choices of m and r_0 may be obtained from Fig. 1 and (25).

The values of βr_0 for which $\bar{T}$ assumes its minimum are shown in Fig. 2.

Figure 3 indicates the fraction of the complete kinetic energy which originates in the three-body correction to $g(r)$. We note that the contribution of $g^{(3)}(r)$ is appreciably reduced by the antisymmetrization of ψ, and, therefore, that the cluster development will be applicable to relatively dense Fermi systems.

High Density Limit

Inspection of Fig. 3(a) suggests that the cluster development cannot be applied to a Bose gas at densities above that corresponding to $(r_0/L)\sim0.4$. However, an estimate of the kinetic energy in the high density limit may be obtained from the cell model. For this purpose we choose a body-centered cubic arrangement of lattic spacing a. The minimum separation of particles is $(\sqrt{3}/2)a$, and the number per unit volume, $2/a^3$; a particle in such a lattice has, therefore, a kinetic energy roughly given by

$$\frac{1}{N}\bar{T}=3\frac{\hbar^2}{2m}\frac{\pi^2}{4[(\sqrt{3}/2)a-r_0]^2}=\frac{\hbar^2}{2mr_0^2}\frac{7.4(r_0/L)^2}{(1.1-r_0/L)^2}. \quad (28)$$

As a criterion for the application of (28) we take the condition, $2r_0>a$ or $r_0/L>0.63$, corresponding to densities great enough to prevent the passage of a particle between two neighbors.

The expression (28) is plotted in Fig. 4(a), and the sum of (28) and the Fermi energy (30a) in Fig. 4(b). The same figure also shows the results of the cluster development, taken from Fig. 1. Simple formulas are given below which pass to the correct low density limits and represent both branches within 20 percent over the entire density range:

TABLE I. Hard-sphere Bose gas: dependence of $\bar{T}$ on the form of $f(r)$.

	B	βr_0
(i) Gaussian	2.05	$\sqrt{3}/2$
(ii) Exponential	1.21	$\sqrt{2}/2$
(iii) Yukawa	1.00	$[4\pi(r_0/L)^3]^{\frac{1}{3}}$

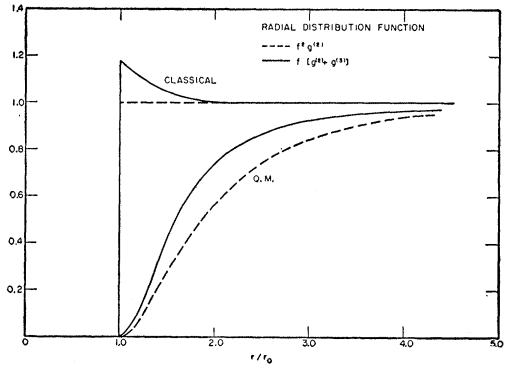

FIG. 5. The pair distribution function, $g(r)$ of the present calculation, compared with the classical result.

Fermions: $$T=\frac{\hbar^2}{2mr_0^2}\frac{9.2(r_0/L)^2}{(1.1-r_0/L)^2};$$

Bosons: $$T=\frac{\hbar^2}{2mr_0^2}\frac{14.7(r_0/L)^3}{(1.1-r_0/L)^2}.$$

Radial Distribution Functions

The distribution functions $g(r)$ of the classical[6] and quantum-mechanical (Bose) hard-sphere gases are compared in Figs. 5 and 6 for $r_0/L=0.5$. Figure 6 indicates that the three-body correction to $g(r)$ in the quantum gas is substantially larger than its classical analog. In this connection we note that for fixed density the expansion parameter of the classical cluster development is the radius of interaction (r_0), while the corresponding parameter of the quantum mechanical development is the generally greater radius of distortion ($\sim r_0+1/\beta$) of the two-particle wave function.

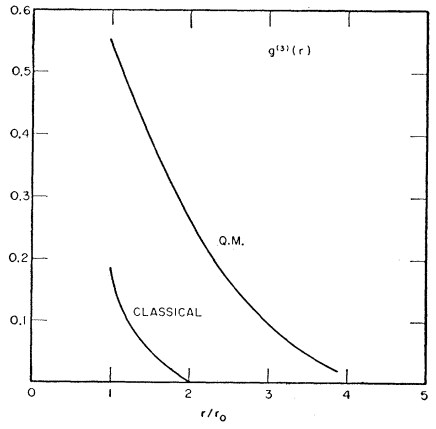

FIG. 6. The three-body correction, $g^{(3)}(r)$ to the pair distribution function.

[6] J. G. Kirkwood, J. Chem. Phys. 5, 67 (1935).

APPENDIX

The density expansion of $\bar{H}$, for a hard sphere gas of Fermi particles, follows from the expansion of (18), (22), and (23) in powers of $(k_f r_{ij})^2$:

$$l(k_f r)=1-(k_f r)^2/10+(k_f r)^4/280-\cdots,$$

$$Q_{\alpha_i\alpha_j}=[(k_{\alpha i}-k_{\alpha j})^2/6]\int r^2 h(r)d^3r - [(k_{\alpha i}-k_{\alpha j})^4/120]\int r^4 h(r)d^3r+\cdots.$$

Inserting these expressions in (21)–(24) and carrying out the sums over k-space, we obtain the following series in n:

$$(2m/\hbar^2)(\bar{T}_c/N)=n\int d^3r_{12}f^2(r_{12})[V^2 f(r_{12})/f(r_{12})] \times[(6\pi^2/5)^{\frac{2}{3}}r_{12}^2 n^{\frac{2}{3}}-(3/175)(6\pi^2)^{4/3}r^4 n^{4/3}+\cdots] \quad (29a)$$

$$+n^2\int d^3r_{12}d^3r_{13}f^2(r_{12})[\nabla^2 f(r_{12})/f(r_{12})]h(r_{13})h(r_{23}) \times[(6\pi^2)^{4/3}n^{4/3}(1/280-3/175)(r_{12}^4+r_{13}^4+r_{23}^4) +(2/100)(r_{12}^2r_{13}^2+r_{12}^2r_{23}^2+r_{13}^2r_{23}^2)+\cdots]. \quad (29b)$$

$$(2m/\hbar^2)(\bar{T}_F/N)=(3/5)(6\pi^2)^{\frac{2}{3}}n^{\frac{2}{3}} \quad (30a)$$

$$+6(6\pi^2)^{4/3}(1/50-23/1050)n^{7/3}\int h(r)d^3r+\cdots, \quad (30b)$$

$$(2m/\hbar^2)(\bar{T}_{FC}/N)=(6/5)(6\pi^2)^{\frac{2}{3}}n^{\frac{2}{3}} \times\int r_{12}f(r_{12})f'(r_{12})d^3r_{12}+\cdots, \quad (31)$$

$\bar{V}/N$: replace $(\hbar^2/2m)(\nabla^2 f/f)$ by V in (33a). (32)

PHYSICAL REVIEW VOLUME 98, NUMBER 5 JUNE 1, 1955

Statistical Model for High-Energy Events*

JOSEPH V. LEPORE AND MAURICE NEUMAN
Radiation Laboratory, Department of Physics, University of California, Berkeley, California

(Received July 16, 1954)

The relative probabilities for alternate processes initiated by a nucleon-nucleon collision depend on the dynamics involved and on the volume in phase space accessible to each final state. The assignment of relative *a priori* probabilities to the final states proportional to their extension in phase must be consistent with the translational, rotational, and Lorentz-invariant properties of the colliding system. The latter in particular implies a conservation law for the center of energy. Its effect is not only to lower the power of the configurational volume by one dimension but also to severely reduce the contributions from high momenta to the phase space integrals.

The limitations on accessibility arising from the controllable constants of motion are not sufficient to insure well defined probabilities. Some additional restriction on the configurational part of the phase space must be imposed. A cutoff factor for *each* particle is accordingly introduced. The configurational volume accessible to the particle thus decreases with increasing energy, a picture not inconsistent with the uncertainty principle.

THIS note describes a statistical model which differs in some essential respects from the one proposed by Fermi.[1]

Following Fermi we assume that in a high-energy collision a state approximating that of equilibrium is established. The probability of disintegration into various possible modes is then taken proportional to their relative extensions in accessible phase space. The limitations on accessibility arise from the assumed controllable constants of motion.

In this note they are taken to be energy, momentum, center of energy (the relativistic analog of center of mass), and isotopic spin. For simplicity conservation of angular momentum has been neglected.

If it is assumed that the extension in accessible phase space (in the center-of-momentum system) corresponding to particles of masses M_1, M_2, $\cdots$ can be approximated by the classical phase integral divided by $h^{3(n-1)}$,

$$P_n=\frac{S_n T_n}{(2\pi\hbar)^{3(n-1)}}\int\prod_{i=1}^{i=n}d\mathbf{p}_i d\mathbf{x}_i\delta(E-\sum_i E_i) \times\delta(-\sum\mathbf{p}_i)\delta\left(\frac{\sum\mathbf{x}_i E_i}{E}\right), \quad (1)$$

one sees immediately that this integral does not converge and therefore some additional restriction is necessary to give the phase integral a well-defined meaning. This difficulty is overcome in the quantum theory by enclosing the system in a container whose walls aer eventually removed to infinity since (having been

* This work was performed under the auspices of the U. S. Atomic Energy Commission.

[1] E. Fermi, Progr. Theoret. Phys. (Japan) **5**, 4, 570 (1950).